United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Friday, 27 November 2015

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers, set squares and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.*
- *Staple all the pages neatly together in the top* left *hand corner.*
- *To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until 8am GMT on Saturday 28 November.*

Do not turn over until **told to do so.**

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1. On Thursday 1st January 2015, Anna buys one book and one shelf. For the next two years, she buys one book every day and one shelf on alternate Thursdays, so she next buys a shelf on 15th January 2015. On how many days in the period Thursday 1st January 2015 until (and including) Saturday 31st December 2016 is it possible for Anna to put all her books on all her shelves, so that there is an equal number of books on each shelf?

2. Let $ABCD$ be a cyclic quadrilateral and let the lines CD and BA meet at E. The line through D which is tangent to the circle ADE meets the line CB at F. Prove that the triangle CDF is isosceles.

3. Suppose that a sequence $t_0, t_1, t_2, \ldots$ is defined by a formula $t_n = An^2 + Bn + C$ for all integers $n \geq 0$. Here A, B and C are real constants with $A \neq 0$. Determine values of A, B and C which give the greatest possible number of successive terms of the sequence which are also successive terms of the Fibonacci sequence. *The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$ and $F_m = F_{m-1} + F_{m-2}$ for $m \geq 2$.*

4. James has a red jar, a blue jar and a pile of 100 pebbles. Initially both jars are empty. A move consists of moving a pebble from the pile into one of the jars or returning a pebble from one of the jars to the pile. The numbers of pebbles in the red and blue jars determine the *state* of the game. The following conditions must be satisfied:

   a) The red jar may never contain fewer pebbles than the blue jar;
   b) The game may never be returned to a previous state.

   What is the maximum number of moves that James can make?

5. Let ABC be a triangle, and let D, E and F be the feet of the perpendiculars from A, B and C to BC, CA and AB respectively. Let P, Q, R and S be the feet of the perpendiculars from D to BA, BE, CF and CA respectively. Prove that P, Q, R and S are collinear.

6. A positive integer is called *charming* if it is equal to 2 or is of the form $3^i 5^j$ where i and j are non-negative integers. Prove that every positive integer can be written as a sum of different charming integers.